The Drinfel'd polynomial of a tridiagonal pair

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Dedicated to Dijen Ray-Chaudhuri on his 75th birthday

Abstract

Let $\mathbb{F}$ denote a field and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions: (i) each of A, A^* is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there is no subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a *tridiagonal pair* on V. It is known that $d = \delta$ and for $0 \leq i \leq d$ the dimensions of V_i, V_{d-i}, V_i^*, V_{d-i}^* coincide. The pair A, A^* is called *sharp* whenever $\dim V_0 = 1$. It is known that if $\mathbb{F}$ is algebraically closed then A, A^* is sharp. Assuming A, A^* is sharp, we use the data $\Phi = (A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$ to define a polynomial P in one variable and degree at most d. We show that P remains invariant if Φ is replaced by $(A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$ or $(A; \{V_i\}_{i=0}^d; A^*; \{V_{d-i}^*\}_{i=0}^d)$ or $(A^*; \{V_i^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d)$. We call P the *Drinfel'd polynomial* of A, A^*. We explain how P is related to the classical Drinfel'd polynomial from the theory of Lie algebras and quantum groups. We expect that the roots of P will be useful in a future classification of the sharp tridiagonal pairs. We compute the roots of P for the case in which V_i and V_i^* have dimension 1 for $0 \leq i \leq d$.

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1 Tridiagonal pairs

Throughout this paper $\mathbb{F}$ denotes a field and V denotes a vector space over $\mathbb{F}$ with finite positive dimension.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we call W an *eigenspace* of A

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whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ is the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1 [7, Definition 1.1] By a *tridiagonal pair* on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta), \tag{2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair A, A^* is *over* $\mathbb{F}$.

Note 1.2 According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a tridiagonal pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

We refer the reader to [1, 2, 8–10, 14, 21, 27, 34] and the references therein for background on tridiagonal pairs.

In order to motivate our results we recall a few facts about tridiagonal pairs. Let A, A^* denote a tridiagonal pair on V, as in Definition 1.1. By [7, Lemma 4.5] the integers d and δ from (ii), (iii) are equal; we call this common value the *diameter* of the pair. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. Then the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). By [7, Corollary 5.7], for $0 \leq i \leq d$ the spaces V_i, V_i^* have the same dimension; we denote this common dimension by ρ_i. By [7, Corollaries 5.7, 6.6] the sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of A, A^*. We say A, A^* is *sharp* whenever $\rho_0 = 1$. By [24, Theorem 1.3], if $\mathbb{F}$ is algebraically closed then A, A^* is sharp. It is an open problem to classify the sharp tridiagonal pairs up to isomorphism [25]. By a *Leonard pair* we mean a tridiagonal pair with shape $(1, 1, \ldots, 1)$ [26, Definition 1.1]. In [26, 32] the Leonard pairs are classified up to isomorphism. This classification yields a correspondence between the

Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials and their relatives [3,32,33]. This family coincides with the terminating branch of the Askey scheme [13]. See [16–20, 29–31, 35] for more information about Leonard pairs.

We now describe our main results. Let A, A^* denote a sharp tridiagonal pair on V. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). Using the data $(A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$ and the following construction, we define a polynomial P in one variable λ. For $0 \le i \le d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*). By [7, Theorem 4.6], for $0 \le i \le d$ the space V_0^* is invariant under

$$(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)(A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I);$$

let ζ_i denote the corresponding eigenvalue. By [7, Theorem 11.1] there exists $\beta \in \mathbb{F}$ such that $\theta_{i-1} - \beta\theta_i + \theta_{i+1}$ and $\theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^*$ are independent of i for $1 \le i \le d-1$. For the moment assume $\beta \ne \pm 2$ and put $q^2 + q^{-2} = \beta$. Then $q^{2i} \ne 1$ for $1 \le i \le d$, by Note 4.8 below. Define

$$P = \sum_{i=0}^{d} \zeta_i p_{i+1} p_{i+2} \cdots p_d,$$

where

$$p_i = (\theta_0\theta_d^* + \theta_d\theta_0^* - \lambda)\frac{(q^i - q^{-i})^2}{(q^d - q^{-d})^2} + (\theta_0 - \theta_i)(\theta_0^* - \theta_i^*)$$

for $1 \le i \le d$. For $\beta = \pm 2$ our definition of P is slightly different; see Definition 7.1 and Definition 8.1 for the details. For all β we show that P remains invariant if the data $(A; \{V_i\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$ is replaced by $(A; \{V_{d-i}\}_{i=0}^d; A^*; \{V_i^*\}_{i=0}^d)$ or $(A; \{V_i\}_{i=0}^d; A^*; \{V_{d-i}^*\}_{i=0}^d)$ or $(A^*; \{V_i^*\}_{i=0}^d; A; \{V_i\}_{i=0}^d)$. We call P the *Drinfel'd polynomial* of A, A^*. We explain how P is related to the classical Drinfel'd polynomial from the theory of Lie algebras and quantum groups. We expect that the roots of P will be useful in a future classification of the sharp tridiagonal pairs. We compute the roots of P for the case in which A, A^* is a Leonard pair.

2 Tridiagonal systems

When working with a tridiagonal pair, it is often convenient to consider a closely related object called a tridiagonal system. To define a tridiagonal system, we recall a few concepts from linear algebra. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra of all linear transformations from V to V. Let A denote a diagonalizable element of $\mathrm{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A. For $0 \le i \le d$ define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \ne i$ $(0 \le j \le d)$. Here I denotes the identity of $\mathrm{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $\sum_{i=0}^d E_i = I$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \le i, j \le d)$; (iii) $V_i = E_i V$ $(0 \le i \le d)$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Moreover

$$E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j}. \tag{3}$$

We note that each of $\{E_i\}_{i=0}^d$, $\{A^i\}_{i=0}^d$ is a basis for the $\mathbb{F}$-subalgebra of $\mathrm{End}(V)$ generated by A. Now let A, A^* denote a tridiagonal pair on V. An ordering of the primitive idempotents or eigenvalues of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 2.1 [7, Definition 2.1] By a *tridiagonal system* on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(iii) below.

(i) A, A^* is a tridiagonal pair on V.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

We say Φ is *over* $\mathbb{F}$. We call d the *diameter* of Φ.

The following result is immediate from lines (1), (2) and Definition 2.1.

Lemma 2.2 [23, Lemma 2.5] *Let* $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *denote a tridiagonal system. Then for* $0 \le i, j, k \le d$ *the following* (i), (ii) *hold.*

(i) $E_i^* A^k E_j^* = 0 \quad \textit{if } k < |i-j|.$

(ii) $E_i A^{*k} E_j = 0 \quad \textit{if } k < |i-j|.$

Definition 2.3 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*;\ \{E_i^*\}_{i=0}^d)$ denote a tridiagonal system. By the *associated tridiagonal pair* we mean the pair A, A^*. By the *shape* of Φ we mean the shape of A, A^*. We say Φ is *sharp* whenever A, A^* is sharp. We call Φ a *Leonard system* whenever A, A^* is a Leonard pair.

3 The D_4 action

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a tridiagonal system on V. Then each of the following is a tridiagonal system on V:

$$\begin{aligned}
\Phi^* &:= (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d),\\
\Phi^{\downarrow} &:= (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d),\\
\Phi^{\Downarrow} &:= (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).
\end{aligned}$$

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all tridiagonal systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{4}$$

$$\Downarrow * = * \downarrow, \qquad \downarrow * = * \Downarrow, \qquad \downarrow\Downarrow = \Downarrow\downarrow. \tag{5}$$

The group generated by symbols $*, \downarrow, \Downarrow$ subject to the relations (4), (5) is the dihedral group D_4. We recall that D_4 is the group of symmetries of a square, and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all tridiagonal systems. Two tridiagonal systems will be called *relatives* whenever they are in the same orbit of this D_4 action. The relatives of Φ are as follows:

name	relative
Φ	$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$
$\Phi^{\downarrow}$	$(A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$
$\Phi^{\Downarrow}$	$(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$
$\Phi^{\downarrow\Downarrow}$	$(A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d)$
Φ^*	$(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$
$\Phi^{\downarrow *}$	$(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$
$\Phi^{\Downarrow *}$	$(A^*; \{E_i^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$
$\Phi^{\downarrow\Downarrow *}$	$(A^*; \{E_{d-i}^*\}_{i=0}^d; A; \{E_{d-i}\}_{i=0}^d)$

Definition 3.1 Let Φ denote a tridiagonal system. For $g \in D_4$ and for an object f associated with Φ, we let f^g denote the corresponding object associated with $\Phi^{g^{-1}}$ (we have been using this convention all along; an example is $\theta_i^*(\Phi) = \theta_i(\Phi^*)$). We say f is D_4-*invariant* whenever $f^g = f$ for all $g \in D_4$.

Let Φ denote a tridiagonal system over $\mathbb{F}$. In this paper we associate with Φ a certain polynomial P called the Drinfel'd polynomial, and show that P is D_4-invariant.

For later use we remark that the elements $*, \Downarrow$ together generate D_4.

4 The eigenvalues and dual eigenvalues

In order to develop our theory of the Drinfel'd polynomial, we will need some detailed supporting results concerning three sequences of scalars: the eigenvalue sequence, the dual eigenvalue sequence, and the split sequence. The supporting results are contained in this section and the next two.

Definition 4.1 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a tridiagonal system on V. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the eigenspace E_iV (resp. E_i^*V). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ. We observe that $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $\mathbb{F}$.

Lemma 4.2 [7, Theorem 11.1] *With reference to Definition 4.1, the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \tag{6}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Definition 4.3 Let A, A^* denote a tridiagonal pair over $\mathbb{F}$. We associate with A, A^* a scalar $\beta \in \mathbb{F}$ as follows. If the diameter $d \geq 3$ let $\beta + 1$ denote the common value of (6), where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) is a standard ordering of the eigenvalues of A (resp. A^*). If $d \leq 2$ let β denote any nonzero scalar in $\mathbb{F}$. We call β the *base* of A, A^*. By construction, for $d \geq 3$ the tridiagonal pairs A, A^* and A^*, A have the same base. For $d \leq 2$, we always choose the bases such that A, A^* and A^*, A have the same base.

Definition 4.4 Let A, A^* denote a tridiagonal pair over $\mathbb{F}$, with diameter d and base β. We assign to A, A^* a *type* as follows:

type	description
I	$\beta \neq 2$, $\beta \neq -2$
II	$\beta = 2$, $\mathrm{Char}(\mathbb{F}) \neq 2$
III^+	$\beta = -2$, $\mathrm{Char}(\mathbb{F}) \neq 2$, d even
III^-	$\beta = -2$, $\mathrm{Char}(\mathbb{F}) \neq 2$, d odd
IV	$\beta = 0$, $\mathrm{Char}(\mathbb{F}) = 2$

By [7, Theorem 11.2], if A, A^* has type IV then $d = 3$. We say A, A^* has *type* III whenever A, A^* has type III^+ or III^-.

Definition 4.5 Let Φ denote a tridiagonal system. By the *base* (resp. *type*) of Φ we mean the base (resp. type) of the associated tridiagonal pair. By construction, the base (resp. type) of Φ is D_4-invariant.

Let $\overline{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$.

Lemma 4.6 [7, Theorem 11.2] *With reference to Definition 4.1, assume Φ has type I, and fix a nonzero $q \in \overline{\mathbb{F}}$ such that $q^2 + q^{-2} = \beta$. Then there exists a sequence of scalars a, b, c, a^*, b^*, c^* taken from $\overline{\mathbb{F}}$ such that*

$$\begin{aligned} \theta_i &= a + bq^{2i-d} + cq^{d-2i}, \\ \theta^*_i &= a^* + b^*q^{2i-d} + c^*q^{d-2i} \end{aligned}$$

for $0 \leq i \leq d$. The sequence is uniquely determined by q provided $d \geq 2$.

Lemma 4.7 *With reference to Definition 4.1 and Lemma 4.6, for $0 \leq i, j \leq d$ we have*

$$\begin{aligned} \theta_i - \theta_j &= (q^{i-j} - q^{j-i})(bq^{i+j-d} - cq^{d-i-j}), \\ \theta^*_i - \theta^*_j &= (q^{i-j} - q^{j-i})(b^*q^{i+j-d} - c^*q^{d-i-j}). \end{aligned}$$

Note 4.8 With reference to Definition 4.1 and Lemma 4.6, for $1 \leq i \leq d$ we have $q^{2i} \neq 1$; otherwise $\theta_i = \theta_0$ by Lemma 4.7.

Lemma 4.9 *With reference to Definition 4.1 and Lemma 4.6, for $1 \leq i \leq d$ we have*

$$\frac{(\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i)}{(q^i - q^{-i})^2} = bb^*q^{2i-2d} + cc^*q^{2d-2i} - bc^* - cb^*.$$

Proof: Use Lemma 4.7. □

Lemma 4.10 [7, Theorem 11.2] *With reference to Definition 4.1, assume Φ has type II. Then there exists a sequence of scalars a, b, c, a^*, b^*, c^* taken from $\mathbb{F}$ such that*

$$\begin{aligned}\theta_i &= a + b(i - d/2) + ci(d - i),\\ \theta^*_i &= a^* + b^*(i - d/2) + c^*i(d - i)\end{aligned}$$

for $0 \leq i \leq d$. The sequence is unique provided $d \geq 2$.

Lemma 4.11 *With reference to Definition 4.1 and Lemma 4.10, for $0 \leq i, j \leq d$ we have*

$$\begin{aligned}\theta_i - \theta_j &= (i - j)(b + c(d - i - j)),\\ \theta^*_i - \theta^*_j &= (i - j)(b^* + c^*(d - i - j)).\end{aligned}$$

Note 4.12 With reference to Definition 4.1 and Lemma 4.10, for all primes $p \leq d$ we have $\mathrm{Char}(\mathbb{F}) \neq p$; otherwise $\theta_p = \theta_0$ by Lemma 4.11. Consequently $\mathrm{Char}(\mathbb{F})$ is 0 or an odd prime greater than d.

Note 4.13 With reference to Definition 4.1 and Lemma 4.10, assume $d \geq 1$. Then $b \neq 0$; otherwise $\theta_0 = \theta_d$. Similary $b^* \neq 0$.

Lemma 4.14 *With reference to Definition 4.1 and Lemma 4.10, for $1 \leq i \leq d$ we have*

$$(\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i)i^{-2} = bb^* + (bc^* + cb^*)(d - i) + cc^*(d - i)^2.$$

Proof: Use Lemma 4.11. □

Lemma 4.15 [7, Theorem 11.2] *With reference to Definition 4.1, assume Φ has type III. Then there exists a sequence of scalars a, b, c, a^*, b^*, c^* taken from $\mathbb{F}$ such that*

$$\begin{aligned}\theta_i &= \begin{cases} a + b + c(i - d/2) & \text{if } i \text{ is even,}\\ a - b - c(i - d/2) & \text{if } i \text{ is odd,}\end{cases}\\ \theta^*_i &= \begin{cases} a^* + b^* + c^*(i - d/2) & \text{if } i \text{ is even,}\\ a^* - b^* - c^*(i - d/2) & \text{if } i \text{ is odd}\end{cases}\end{aligned}$$

for $0 \leq i \leq d$. The sequence is unique provided $d \geq 2$.

Lemma 4.16 *With reference to Definition 4.1 and Lemma 4.15, for $0 \leq i, j \leq d$ we have*

$$\begin{aligned}\theta_i - \theta_j &= \begin{cases} (-1)^i c(i - j) & \text{if } i - j \text{ is even,}\\ (-1)^i(2b + c(i + j - d)) & \text{if } i - j \text{ is odd,}\end{cases}\\ \theta^*_i - \theta^*_j &= \begin{cases} (-1)^i c^*(i - j) & \text{if } i - j \text{ is even,}\\ (-1)^i(2b^* + c^*(i + j - d)) & \text{if } i - j \text{ is odd.}\end{cases}\end{aligned}$$

Note 4.17 With reference to Definition 4.1 and Lemma 4.15, for all primes $p \leq d/2$ we have $\mathrm{Char}(\mathbb{F}) \neq p$; otherwise $\theta_{2p} = \theta_0$ by Lemma 4.16. Consequently $\mathrm{Char}(\mathbb{F})$ is 0 or an odd prime greater than $d/2$.

Note 4.18 With reference to Definition 4.1 and Lemma 4.15, assume $d \geq 2$. Then $c \neq 0$; otherwise $\theta_0 = \theta_2$. Similarly $c^* \neq 0$. For $d \leq 1$ we always choose c, c^* to be nonzero.

Note 4.19 With reference to Definition 4.1 and Lemma 4.15, assume d is odd. Then $b \neq 0$; otherwise $\theta_0 = \theta_d$. Similarly $b^* \neq 0$.

Lemma 4.20 *With reference to Definition 4.1 and Lemma 4.15, for $0 \leq i \leq d$ we have*

$$(\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i) = \begin{cases} cc^* i^2 & \text{if } i \text{ is even}, \\ 4bb^* + 2(bc^* + cb^*)(i-d) + cc^*(i-d)^2 & \text{if } i \text{ is odd}. \end{cases}$$

Proof: Use Lemma 4.16. □

Lemma 4.21 [7, Theorem 11.2] *With reference to Definition 4.1, assume Φ has type IV. Then there exists a unique sequence of scalars a, b, c, a^*, b^*, c^* taken from $\mathbb{F}$ such that*

$$\begin{array}{llll} \theta_0 = a, & \theta_1 = b + c, & \theta_2 = a + c, & \theta_3 = b, \\ \theta^*_0 = a^*, & \theta^*_1 = b^* + c^*, & \theta^*_2 = a^* + c^*, & \theta^*_3 = b^*. \end{array}$$

Note 4.22 With reference to Definition 4.1 and Lemma 4.21, each of $a + b, a + b + c, c$ is nonzero since $\{\theta_i\}_{i=0}^3$ are distinct. Similarly each of $a^* + b^*, a^* + b^* + c^*, c^*$ is nonzero.

Lemma 4.23 *With reference to Definition 4.1 and Lemma 4.21,*

$$\begin{aligned} (\theta_0 - \theta_1)(\theta^*_0 - \theta^*_1) &= (a + b + c)(a^* + b^* + c^*), \\ (\theta_0 - \theta_2)(\theta^*_0 - \theta^*_2) &= cc^*, \\ (\theta_0 - \theta_3)(\theta^*_0 - \theta^*_3) &= (a + b)(a^* + b^*). \end{aligned}$$

Proof: Use Lemma 4.21. □

When discussing tridiagonal systems we will use the following notational convention. Let λ denote an indeterminate and let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{F}$. With reference to Definition 4.1, for $0 \leq i \leq d$ we define the following polynomials in $\mathbb{F}[\lambda]$:

$$\begin{aligned} \tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\ \eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\ \tau^*_i &= (\lambda - \theta^*_0)(\lambda - \theta^*_1) \cdots (\lambda - \theta^*_{i-1}), \\ \eta^*_i &= (\lambda - \theta^*_d)(\lambda - \theta^*_{d-1}) \cdots (\lambda - \theta^*_{d-i+1}). \end{aligned}$$

Note that each of τ_i, η_i, τ^*_i, η^*_i is monic with degree i.

Lemma 4.24 *With reference to Definition 4.1, the following* (i)–(iv) *hold for* $0 \leq i, j \leq d$.

(i) $\tau_i(\theta_j) = 0$ *if and only if* $j < i$;

(ii) $\eta_i(\theta_j) = 0$ *if and only if* $j > d - i$;

(iii) $\tau_i^*(\theta_j^*) = 0$ *if and only if* $j < i$;

(iv) $\eta_i^*(\theta_j^*) = 0$ *if and only if* $j > d - i$.

5 Some scalars

Adopt the notation of Definition 4.1. For nonnegative integers r, s, t such that $r+s+t \leq d$, in [28, Definition 13.1] we defined some scalars $[r, s, t]_q \in \mathbb{F}$. By [28, Definition 13.1] these scalars are rational functions of the base β, and in this paper we are going to drop the subscript q altogether and just write $[r, s, t]$. These scalars are described in the next definition. We will use the following notation. For all $a, q \in \overline{\mathbb{F}}$ define

$$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \qquad n = 0, 1, 2 \dots \tag{7}$$

We interpret $(a; q)_0 = 1$.

Definition 5.1 [28, Lemma 13.2] With reference to Definition 4.1, let r, s, t denote nonnegative integers such that $r + s + t \leq d$. We define $[r, s, t]$ as follows.

(i) Assume Φ is type I. Then

$$[r, s, t] = \frac{(q^2; q^2)_{r+s}(q^2; q^2)_{r+t}(q^2; q^2)_{s+t}}{(q^2; q^2)_r(q^2; q^2)_s(q^2; q^2)_t(q^2; q^2)_{r+s+t}}. \tag{8}$$

Here $q^2 + q^{-2} = \beta$ where β is the base of Φ.

(ii) Assume Φ is type II. Then

$$[r, s, t] = \frac{(r+s)!\,(r+t)!\,(s+t)!}{r!\,s!\,t!\,(r+s+t)!}. \tag{9}$$

(iii) Assume Φ is type III. If each of r, s, t is odd, then $[r, s, t] = 0$. If at least one of r, s, t is even, then

$$[r, s, t] = \frac{\lfloor \frac{r+s}{2} \rfloor! \lfloor \frac{r+t}{2} \rfloor! \lfloor \frac{s+t}{2} \rfloor!}{\lfloor \frac{r}{2} \rfloor! \lfloor \frac{s}{2} \rfloor! \lfloor \frac{t}{2} \rfloor! \lfloor \frac{r+s+t}{2} \rfloor!}. \tag{10}$$

The expression $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(iv) Assume Φ is type IV. Recall in this case $d = 3$. If each of r, s, t equals 1, then $[r, s, t] = 0$. If at least one of r, s, t equals 0 then $[r, s, t] = 1$.

We have a comment.

Lemma 5.2 *With reference to Definitions 4.1 and 5.1 the following* (i), (ii) *hold.*

(i) *The expression $[r, s, t]$ is symmetric in r, s, t.*

(ii) *If at least one of r, s, t is zero then $[r, s, t] = 1$.*

(iii) *The expression $[r, s, t]$ is D_4-invariant.*

Lemma 5.3 *With reference to Definition 4.1, let r, s, t, u denote nonnegative integers whose sum is at most d. Then*

$$[r, s, t + u]\ [t, u, r + s] = [s, u, r + t]\ [r, t, s + u].$$

Proof: For each type I–IV this is a routine cancellation using the formulae in Definition 5.1. $\square$

The following lemma shows one significance of the scalars $[r, s, t]$.

Lemma 5.4 [22, Lemma 9.1] With reference to Definition 4.1,

$$\eta_i = \sum_{h=0}^{i} [h, i - h, d - i]\eta_{i-h}(\theta_0)\tau_h \qquad (0 \leq i \leq d).$$

6 The split decomposition

In this section we recall the split decomposition associated with a tridiagonal system [7, Section 4]. With reference to Definition 4.1, for $0 \leq i \leq d$ define

$$U_i = (E^*_0V + E^*_1V + \cdots + E^*_iV) \cap (E_iV + E_{i+1}V + \cdots + E_dV). \tag{11}$$

By [7, Theorem 4.6]

$$V = U_0 + U_1 + \cdots + U_d \qquad \text{(direct sum)},$$

and for $0 \leq i \leq d$ both

$$U_0 + U_1 + \cdots + U_i = E^*_0V + E^*_1V + \cdots + E^*_iV, \tag{12}$$

$$U_i + U_{i+1} + \cdots + U_d = E_iV + E_{i+1}V + \cdots + E_dV. \tag{13}$$

By [7, Corollary 5.7] U_i has dimension ρ_i for $0 \leq i \leq d$, where $\{\rho_i\}_{i=0}^d$ is the shape of Φ. By [7, Theorem 4.6] both

$$(A - \theta_iI)U_i \subseteq U_{i+1}, \tag{14}$$

$$(A^* - \theta^*_iI)U_i \subseteq U_{i-1} \tag{15}$$

for $0 \leq i \leq d$, where $U_{-1} = 0$ and $U_{d+1} = 0$. The sequence $\{U_i\}_{i=0}^d$ is called the Φ-*split decomposition* of V [7, Section 4]. Now assume that Φ is sharp, so that U_0 has dimension 1. By (14), (15), for $0 \leq i \leq d$ the space U_0 is invariant under

$$(A^* - \theta^*_1I)(A^* - \theta^*_2I)\cdots(A^* - \theta^*_iI)(A - \theta_{i-1}I)\cdots(A - \theta_1I)(A - \theta_0I); \tag{16}$$

let ζ_i denote the corresponding eigenvalue. Note that $\zeta_0 = 1$. We call the sequence $\{\zeta_i\}_{i=0}^d$ the *split sequence* of Φ.

Note 6.1 In the literature on Leonard systems there are two sequences of scalars called the first split sequence and the second split sequence [26, Section 3]. These sequences are related to the above split sequence as follows. With reference to Definition 4.1, assume Φ is a Leonard system and let $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denote the first split sequence (resp. second split sequence) for Φ in the sense of [26, Section 3]. Then the sequence $\{\varphi_1\varphi_2\cdots\varphi_i\}_{i=0}^d$ (resp. $\{\phi_1\phi_2\cdots\phi_i\}_{i=0}^d$) is the split sequence of Φ (resp. $\Phi^{\Downarrow}$).

The D_4 action affects the split sequence as follows.

Lemma 6.2 [22, Theorem 7.3] *With reference to Definition 4.1 assume Φ is sharp. Then*

$$\zeta_i^* = \zeta_i \qquad\qquad 0 \le i \le d.$$

Lemma 6.3 [22, Theorem 9.3] *With reference to Definition 4.1 assume Φ is sharp. Then*

$$\zeta_i^{\Downarrow} = \sum_{h=0}^{i} [h, i-h, d-i] \frac{\eta_{d-h}^*(\theta_0^*)\eta_{i-h}(\theta_0)}{\eta_{d-i}^*(\theta_0^*)}\, \zeta_h \qquad\qquad 0 \le i \le d.$$

Definition 6.4 [22, Definition 6.2] Let Φ denote a sharp tridiagonal system. By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ and $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ.

Proposition 6.5 [24, Theorem 1.6] *Two sharp tridiagonal systems over $\mathbb{F}$ are isomorphic if and only if they have the same parameter array.*

7 The Drinfel'd polynomial

Let Φ denote a sharp tridiagonal system. In this section we introduce the Drinfel'd polynomial of Φ, for all types except III^+. In Section 8 we will treat type III^+.

Definition 7.1 Let Φ denote a sharp tridiagonal system over $\mathbb{F}$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Assume Φ is not type III^+. For $1 \le i \le d$ we define a polynomial $p_i \in \mathbb{F}[\lambda]$ by

$$p_i = (\theta_0\theta_d^* + \theta_d\theta_0^* - \lambda)\alpha_i + (\theta_0 - \theta_i)(\theta_0^* - \theta_i^*), \tag{17}$$

where

Case	I	II	III^- (i even)	III^- (i odd)	IV (i even)	IV (i odd)
α_i	$\frac{(q^i-q^{-i})^2}{(q^d-q^{-d})^2}$	$\frac{i^2}{d^2}$	0	1	0	1

For type I, $q^2 + q^{-2} = \beta$ where β is the base of Φ from Definition 4.5.

Note 7.2 Referring to the table in Definition 7.1, for $1 \le i \le d$ the expression α_i is a rational function of the base β. For example if $d = 3$,

$$\alpha_1 = (\beta+1)^{-2}, \qquad\qquad \alpha_2 = (\beta+2)(\beta+1)^{-2}, \qquad\qquad \alpha_3 = 1.$$

Lemma 7.3 *With reference to Definition 7.1 the following* (i), (ii) *hold for* $1 \leq i \leq d$.

(i) α_i *is* D_4*-invariant.*

(ii) $p_i^* = p_i$.

Proof: (i) Follows from the table in Definition 7.1, and since β is D_4-invariant.
(ii) Immediate from (17) and (i) above. $\square$

Lemma 7.4 *With reference to Definition 7.1, for* $1 \leq i \leq d$ *both*

$$\begin{aligned} p_i(\theta_0\theta_d^* + \theta_d\theta_0^*) &= (\theta_0 - \theta_i)(\theta_0^* - \theta_i^*), \\ p_i(\theta_0\theta_0^* + \theta_d\theta_d^*) &= (\theta_0 - \theta_i)(\theta_0^* - \theta_i^*) - \alpha_i(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*). \end{aligned}$$

Lemma 7.5 *With reference to Definition 7.1,*

$$p_i^{\Downarrow} = (\theta_0\theta_0^* + \theta_d\theta_d^* - \lambda)\alpha_i + (\theta_d - \theta_{d-i})(\theta_0^* - \theta_i^*) \qquad (1 \leq i \leq d). \tag{18}$$

Proof: Apply $\Downarrow$ to (17) and evaluate the result using Lemma 7.3(i) and $\theta_j^{\Downarrow} = \theta_{d-j}$ $(0 \leq j \leq d)$. $\square$

Lemma 7.6 *With reference to Definition 7.1, for* $1 \leq i \leq d$ *both*

$$\begin{aligned} p_i^{\Downarrow}(\theta_0\theta_0^* + \theta_d\theta_d^*) &= (\theta_d - \theta_{d-i})(\theta_0^* - \theta_i^*), \\ p_i^{\Downarrow}(\theta_0\theta_d^* + \theta_d\theta_0^*) &= (\theta_d - \theta_{d-i})(\theta_0^* - \theta_i^*) + \alpha_i(\theta_0 - \theta_d)(\theta_0^* - \theta_d^*). \end{aligned}$$

Lemma 7.7 *With reference to Definition 7.1, for* $d \geq 1$ *both*

$$\begin{aligned} p_d &= \theta_0\theta_0^* + \theta_d\theta_d^* - \lambda, \\ p_d^{\Downarrow} &= \theta_0\theta_d^* + \theta_d\theta_0^* - \lambda. \end{aligned}$$

Proof: Set $i = d$ in (17), (18) and observe that $\alpha_d = 1$. $\square$

Definition 7.8 With reference to Definition 7.1, we define a polynomial $P \in \mathbb{F}[\lambda]$ by

$$P = \sum_{i=0}^{d} \zeta_i p_{i+1} p_{i+2} \cdots p_d, \tag{19}$$

where $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ from Section 6. We call P the *Drinfel'd polynomial* of Φ.

Example 7.9 With reference to Definition 7.1 and Definition 7.8, if $d = 0$ then $P = 1$, and if $d = 1$ then $P = \zeta_1 + \theta_0\theta_0^* + \theta_1\theta_1^* - \lambda$.

With reference to Definition 7.1 and Definition 7.8, we are going to show that P is D_4-invariant. We will prove this after a few lemmas.

Lemma 7.10 *With reference to Definitions 7.1 and 7.8, we have $P^* = P$.*

Proof: Apply $*$ to (19) and evaluate the result using Lemma 6.2 and Lemma 7.3(ii). $\square$

Lemma 7.11 *With reference to Definition 7.1, for $1 \le i \le j \le d$ we have*

$$[i, j-i, d-j]\alpha_i = [i-1, j-i, d-j+1]\alpha_j a_{i,j} \tag{20}$$

where $a_{i,j}$ is given below.

type of Φ	I	II	IV
$a_{i,j}$	$\frac{(q^i-q^{-i})(q^{d-j+1}-q^{j-d-1})}{(q^j-q^{-j})(q^{d-i+1}-q^{i-d-1})}$	$\frac{i(d-j+1)}{j(d-i+1)}$	1

For type III$^-$ the scalar $a_{i,j}$ depends on the parity of i, j as follows:

Parity of i, j	i even, j even	i odd, j even	i even, j odd	i odd, j odd
$a_{i,j}$	$\frac{i(d-j+1)}{j(d-i+1)}$	$\frac{j-d-1}{j}$	$\frac{i}{i-d-1}$	1

Proof: For each of the above seven entries one routinely verifies (20) using Definition 5.1 and the table in Definition 7.1. $\square$

Lemma 7.12 *With reference to Definition 7.1, for $1 \le i \le j \le d$ we have*

$$[i, j-i, d-j]p_i^{\Downarrow} \quad = \quad [i-1, j-i, d-j+1]a_{i,j}p_j \; + \; c_{i,j} \tag{21}$$

where $a_{i,j}$ is from Lemma 7.11 and

$$\begin{aligned} c_{i,j} \;=\; & [i, j-i, d-j](\theta_d - \theta_{d-i})(\theta_0^* - \theta_i^*) \\ & \qquad - \quad [i-1, j-i, d-j+1]a_{i,j}p_j(\theta_0\theta_0^* + \theta_d\theta_d^*). \end{aligned}$$

Proof: Each side of (21) is a polynomial in $\mathbb{F}[\lambda]$ with degree at most 1. On each side of (21) the λ-coefficients agree by (17), (18) and Lemma 7.11. To see that the constant terms also agree, evaluate each side of (21) at $\lambda = \theta_0\theta_0^* + \theta_d\theta_d^*$ and use the first equation of Lemma 7.6. $\square$

Lemma 7.13 *With reference to Definition 7.1, for $0 \le i \le d$ we have*

$$p_{i+1}^{\Downarrow}p_{i+2}^{\Downarrow}\cdots p_d^{\Downarrow} \quad = \quad \sum_{h=i}^{d}[i, h-i, d-h]\frac{\eta_{d-i}^*(\theta_0^*)\tau_{h-i}(\theta_d)}{\eta_{d-h}^*(\theta_0^*)}p_{h+1}p_{h+2}\cdots p_d.$$

Proof: The proof is by induction on $i = d, d-1, \ldots, 0$. Let i be given. The assertion for $i = d$ holds trivially, so assume $i < d$. By induction,

$$p^{\Downarrow}_{i+2} \cdots p^{\Downarrow}_{d} \;=\; \sum_{h=i+1}^{d} [i+1, h-i-1, d-h] \frac{\eta^*_{d-i-1}(\theta^*_0)\tau_{h-i-1}(\theta_d)}{\eta^*_{d-h}(\theta^*_0)} p_{h+1} \cdots p_d.$$

In this equation we multiply both sides by $p^{\Downarrow}_{i+1}$ and use Lemma 7.12 to get

$$\begin{aligned} p^{\Downarrow}_{i+1} p^{\Downarrow}_{i+2} \cdots p^{\Downarrow}_{d} \;&=\; \sum_{h=i+1}^{d} \Big([i, h-i-1, d-h+1] a_{i+1,h} p_h + c_{i+1,h} \Big) \\ &\qquad \times \; \frac{\eta^*_{d-i-1}(\theta^*_0)\tau_{h-i-1}(\theta_d)}{\eta^*_{d-h}(\theta^*_0)} p_{h+1} \cdots p_d. \end{aligned}$$

In the sum on the right in the above equation, we collect terms to get a linear combination of $\{p_{h+1} \cdots p_d\}_{h=i}^{d}$. In this linear combination, for $i \le h \le d$ let γ_h denote the coefficient of $p_{h+1} \cdots p_d$. We show

$$\gamma_h = [i, h-i, d-h] \frac{\eta^*_{d-i}(\theta^*_0)\tau_{h-i}(\theta_d)}{\eta^*_{d-h}(\theta^*_0)}. \tag{22}$$

First assume $h = i$. Then line (22) holds since both sides equal 1. Next assume $i+1 \le h \le d-1$. By construction

$$\begin{aligned} \gamma_h \;&=\; [i, h-i, d-h] a_{i+1,h+1} \frac{\eta^*_{d-i-1}(\theta^*_0)\tau_{h-i}(\theta_d)}{\eta^*_{d-h-1}(\theta^*_0)} \\ &+\; c_{i+1,h} \frac{\eta^*_{d-i-1}(\theta^*_0)\tau_{h-i-1}(\theta_d)}{\eta^*_{d-h}(\theta^*_0)}. \end{aligned}$$

Line (22) will follow from this provided that

$$c_{i+1,h} = [i, h-i, d-h](\theta_d - \theta_{h-i-1})\big(\theta^*_0 - \theta^*_{i+1} - a_{i+1,h+1}(\theta^*_0 - \theta^*_{h+1})\big).$$

The above equation is routinely checked using Definition 5.1 and the definition of $c_{i+1,h}$ in Lemma 7.12. We have now verified (22) for $i+1 \le h \le d-1$. Next assume $h = d$. By construction

$$\gamma_d = c_{i+1,d} \eta^*_{d-i-1}(\theta^*_0)\tau_{d-i-1}(\theta_d).$$

Using the definition of $c_{i+1,d}$ in Lemma 7.12 along with Lemma 5.2(ii) and $p_d(\theta_0\theta^*_0 + \theta_d\theta^*_d) = 0$, we find $c_{i+1,d} = (\theta_d - \theta_{d-i-1})(\theta^*_0 - \theta^*_{i+1})$. By these comments

$$\gamma_d = \eta^*_{d-i}(\theta^*_0)\tau_{d-i}(\theta_d).$$

By the above line and Lemma 5.2(ii), line (22) holds at $h = d$. We have now verified (22) for $i \le h \le d$ and the result follows. □

Lemma 7.14 *With reference to Definitions 7.1 and 7.8, we have $P^{\Downarrow} = P$.*

Proof: By Definition 7.8,

$$P^{\Downarrow} = \sum_{i=0}^{d} \zeta_i^{\Downarrow} p_{i+1}^{\Downarrow} p_{i+2}^{\Downarrow} \cdots p_d^{\Downarrow}. \tag{23}$$

In (23), for $0 \leq i \leq d$ we expand $\zeta_i^{\Downarrow}$ and $p_{i+1}^{\Downarrow} p_{i+2}^{\Downarrow} \cdots p_d^{\Downarrow}$ using Lemma 6.3 and Lemma 7.13, respectively. Now $P^{\Downarrow}$ becomes a linear combination of terms, each of the form $\zeta_r p_{s+1} p_{s+2} \cdots p_d$ with $0 \leq r \leq s \leq d$. To show $P^{\Downarrow} = P$, we show that for $0 \leq r \leq s \leq d$ the coefficient of $\zeta_r p_{s+1} p_{s+2} \cdots p_d$ in this linear combination is $\delta_{r,s}$. Using in order (23), Lemma 6.3, Lemma 7.13, a change of variables $h = s-i$, Lemma 5.3, Lemma 5.4, Lemma 4.24(ii), Lemma 5.2(ii), the coefficient is

$$\begin{aligned}
&\sum_{i=r}^{s} [r, i-r, d-i] \frac{\eta^*_{d-r}(\theta^*_0)\eta_{i-r}(\theta_0)}{\eta^*_{d-i}(\theta^*_0)} [i, s-i, d-s] \frac{\eta^*_{d-i}(\theta^*_0)\tau_{s-i}(\theta_d)}{\eta^*_{d-s}(\theta^*_0)} \\
&= \frac{\eta^*_{d-r}(\theta^*_0)}{\eta^*_{d-s}(\theta^*_0)} \sum_{i=r}^{s} [r, i-r, d-i][i, s-i, d-s] \eta_{i-r}(\theta_0)\tau_{s-i}(\theta_d) \\
&= \frac{\eta^*_{d-r}(\theta^*_0)}{\eta^*_{d-s}(\theta^*_0)} \sum_{h=0}^{s-r} [r, s-r-h, d-s+h][s-h, h, d-s] \eta_{s-r-h}(\theta_0)\tau_h(\theta_d) \\
&= \frac{\eta^*_{d-r}(\theta^*_0)}{\eta^*_{d-s}(\theta^*_0)} [r, s-r, d-s] \sum_{h=0}^{s-r} [h, s-r-h, d-s+r] \eta_{s-r-h}(\theta_0)\tau_h(\theta_d) \\
&= \frac{\eta^*_{d-r}(\theta^*_0)}{\eta^*_{d-s}(\theta^*_0)} [r, s-r, d-s] \eta_{s-r}(\theta_d) \\
&= \delta_{r,s}.
\end{aligned}$$

The result follows. $\square$

Proposition 7.15 *With reference to Definitions 7.1 and 7.8, the polynomial P is D_4-invariant.*

Proof: This follows from Lemma 7.10 and Lemma 7.14, since D_4 is generated by $*, \Downarrow$. $\square$

Proposition 7.16 *With reference to Definitions 7.1 and 7.8, the following* (i), (ii) *hold.*

(i) $P(\theta_0\theta^*_0 + \theta_d\theta^*_d) = \zeta_d$;

(ii) $P(\theta_0\theta^*_d + \theta_d\theta^*_0) = \zeta_d^{\Downarrow}$.

Proof: (i) Assume $d \geq 1$; otherwise both sides equal 1. Now evaluate (19) at $\lambda = \theta_0\theta^*_0 + \theta_d\theta^*_d$ and observe $p_d(\theta_0\theta^*_0 + \theta_d\theta^*_d) = 0$ by Lemma 7.7.
(ii) Apply (i) to $\Phi^{\Downarrow}$. $\square$

8 The Drinfel'd polynomial for type III^+

In this section we define the Drinfel'd polynomial for a sharp tridiagonal system of type III^+. In order to motivate our definition, consider the type I expression for α_i that appears in the table of Definition 7.1. That expression is a fraction with $(q^i - q^{-i})^2$ in the numerator and $(q^d - q^{-d})^2$ in the denominator. Recall that $q^2 + q^{-2} = \beta$ where β is the type of Φ. Under the assumption of type III^+ we have $\beta = -2$ so q^2 becomes -1. In this case the expression $(q^i - q^{-i})^2$ becomes 0 if i is even and -4 if i is odd. Since d is even the expression $(q^d - q^{-d})^2$ becomes 0. For i even we can "take limits" and evaluate α_i in a reasonable way, but if we try this for i odd then α_i becomes ∞. To resolve this problem we multiply the polynomial P in Definition 7.8 by $(q^d - q^{-d})^d$ before taking limits. This procedure gives a reasonable limit but it does have the effect of sending the i-summand in (19) to zero for $1 \leq i \leq d$. Consequently for type III^+ the Drinfel'd polynomial is no longer defined as a sum. Instead it looks as follows.

Definition 8.1 Let Φ denote a sharp tridiagonal system over $\mathbb{F}$ with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$. Assume Φ has type III^+. We define the Drinfel'd polynomial of Φ to be

$$P = p_1 p_2 \cdots p_d \tag{24}$$

where for $1 \leq i \leq d$,

$$p_i = \begin{cases} (\theta_0\theta^*_d + \theta_d\theta^*_0 - \lambda)i^2/d^2 + (\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i) & \text{if } i \text{ is even,} \\ \theta_0\theta^*_d + \theta_d\theta^*_0 - \lambda & \text{if } i \text{ is odd.} \end{cases} \tag{25}$$

Note that $P = 1$ if $d = 0$.

Referring to Definition 8.1, we now put P in a more attractive form.

Lemma 8.2 *With reference to Definition 8.1, for even i $(1 \leq i \leq d)$ we have*

$$p_i = (\theta_0\theta^*_0 + \theta_d\theta^*_d - \lambda)i^2/d^2.$$

Proof: By Lemma 4.20,

$$(\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i) = (\theta_0 - \theta_d)(\theta^*_0 - \theta^*_d)i^2/d^2.$$

Evaluating the first line of (25) using this we get the result. $\square$

Lemma 8.3 *With reference to Definition 8.1 and in the notation of Lemma 4.15, for $1 \leq i \leq d$ we have*

$$p_i = \begin{cases} (2(a+b)(a^*+b^*) + cc^*d^2/2 - \lambda)i^2/d^2 & \text{if } i \text{ is even,} \\ 2(a+b)(a^*+b^*) - cc^*d^2/2 - \lambda & \text{if } i \text{ is odd.} \end{cases}$$

Proof: Evaluate (25) using Lemma 4.15 and Lemma 8.2. □

Lemma 8.4 *With reference to Definition 8.1, assume $d \geq 2$ and abbreviate $D = d/2$.*

(i) *P looks as follows in terms of the eigenvalues and dual eigenvalues:*

$$P \;=\; D^{-d}\,(D!)^2\,(\theta_0\theta^*_0 + \theta_d\theta^*_d - \lambda)^{d/2}\,(\theta_0\theta^*_d + \theta_d\theta^*_0 - \lambda)^{d/2}.$$

(ii) *P looks as follows in the notation of Lemma 4.15:*

$$P \;=\; D^{-d}\,(D!)^2\,\big(2(a+b)(a^*+b^*) + cc^*d^2/2 - \lambda\big)^{d/2} \\ \times\quad \big(2(a+b)(a^*+b^*) - cc^*d^2/2 - \lambda\big)^{d/2}.$$

Proof: (i) Evaluate (24) using (25) and Lemma 8.2.
(ii) Evaluate (24) using Lemma 8.3. □

Proposition 8.5 *With reference to Definition 8.1, P is D_4-invariant.*

Proof: Use Lemma 8.4(i). □

Proposition 8.6 *With reference to Definition 8.1, the following* (i), (ii) *hold for $d \geq 2$.*

(i) $P(\theta_0\theta^*_0 + \theta_d\theta^*_d) = 0$;

(ii) $P(\theta_0\theta^*_d + \theta_d\theta^*_0) = 0$.

Proof: Immediate from Lemma 8.4(i). □

Combining Proposition 7.15 and Proposition 8.5 we obtain the following theorem.

Theorem 8.7 *Let Φ denote a sharp tridiagonal system and let P denote the corresponding Drinfel'd polynomial from Definition 7.8 and Definition 8.1. Then P is D_4-invariant.*

Definition 8.8 Let A, A^* denote a sharp tridiagonal pair. By the *Drinfel'd polynomial* of A, A^* we mean the Drinfel'd polynomial of an associated tridiagonal system. By construction A, A^* and A^*, A have the same Drinfel'd polynomial.

9 The normalized Drinfel'd polynomial for type I

Let Φ denote a sharp tridiagonal system over $\mathbb{F}$. In Definition 7.8 and Definition 8.1 we defined the Drinfel'd polynomial P of Φ. One advantage of our definition is that it depends in only a minor way on the type of Φ. One disadvantage is that the roots of P are not as nice as they could be. We make the roots nicer by introducing the *normalized Drinfel'd polynomial* $\hat{P}$ on a type-by-type basis. For each type, $\hat{P}$ is related to P by an equation $\hat{P}(\lambda) = P(u\lambda + v)$, where u, v are scalars in $\overline{\mathbb{F}}$ that depend on the type. For the rest of this section we focus on type I.

Assumption 9.1 Let Φ denote a sharp tridiagonal system over $\mathbb{F}$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$. Assume Φ has type I and let the scalars $q, a, b, c, a^*, b^*, c^*$ be as in Lemma 4.6.

The following definition is motivated by Lemma 4.9.

Definition 9.2 With reference to Assumption 9.1, for $1 \leq i \leq d$ define $\hat{p}_i \in \mathbb{F}[\lambda]$ by

$$\hat{p}_i = (q^i - q^{-i})^2 (bb^* q^{2i-2d} + cc^* q^{2d-2i} - \lambda). \tag{26}$$

By Lemma 4.9,

$$\hat{p}_i (bc^* + cb^*) = (\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i). \tag{27}$$

Definition 9.3 With reference to Assumption 9.1, define $\hat{P} \in \mathbb{F}[\lambda]$ by

$$\hat{P} = \sum_{i=0}^{d} \zeta_i \hat{p}_{i+1} \hat{p}_{i+2} \cdots \hat{p}_d, \tag{28}$$

where $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ and the $\hat{p}_i$ are from Definition 9.2. We call $\hat{P}$ the *normalized Drinfel'd polynomial* for Φ.

Lemma 9.4 *With reference to Assumption 9.1, both*

$$\begin{aligned} u(bb^* + cc^*) + v &= \theta_0 \theta^*_0 + \theta_d \theta^*_d, \\ u(bc^* + cb^*) + v &= \theta_0 \theta^*_d + \theta_d \theta^*_0, \end{aligned}$$

where

$$\begin{aligned} u &= (q^d - q^{-d})^2, \\ v &= 2aa^* + 2(b+c)(b^* + c^*) + (q^d + q^{-d}) a (b^* + c^*) + (q^d + q^{-d}) a^* (b + c). \end{aligned}$$

Proof: Use Lemma 4.6. $\square$

Theorem 9.5 *With reference to Assumption 9.1,*

$$\hat{p}_i(\lambda) = p_i(u\lambda + v) \qquad (1 \leq i \leq d), \tag{29}$$
$$\hat{P}(\lambda) = P(u\lambda + v), \tag{30}$$

where u, v are from Lemma 9.4.

Proof: Each side of (29) is a polynomial in $\mathbb{F}[\lambda]$ with degree 1. On each side of (29) the polynomial has λ coefficient $-(q^i - q^{-i})^2$. Moreover at $\lambda = bc^* + cb^*$ this polynomial takes the value $(\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i)$. We have verified (29) and (30) follows. $\square$

Proposition 9.6 *With reference to Assumption 9.1 and Definition 9.3, the polynomial $\hat{P}$ is D_4-invariant.*

Proof: Follows from Proposition 7.15 and (30). $\square$

Proposition 9.7 *With reference to Assumption 9.1 and Definition 9.3, the following* (i), (ii) *hold.*

(i) $\hat{P}(bb^* + cc^*) = \zeta_d$;

(ii) $\hat{P}(bc^* + cb^*) = \zeta_d^{\Downarrow}$.

Proof: Combine Proposition 7.16, Lemma 9.4, and (30). $\square$

We now consider the normalized Drinfel'd polynomial for a Leonard system of type I.

Lemma 9.8 [15, Theorem 6.1] *With reference to Assumption 9.1, suppose Φ is a Leonard system. Let $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denote the corresponding first (resp. second) split sequence, from [26, Section 3]. Then there exists $t \in \overline{\mathbb{F}}$ such that for $1 \leq i \leq d$,*

$$\begin{aligned}
\varphi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(t - bb^*q^{2i-d-1} - cc^*q^{d+1-2i}),\\
\phi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(t - cb^*q^{2i-d-1} - bc^*q^{d+1-2i}).
\end{aligned}$$

The above expressions can be factored more completely if $bb^*cc^* \neq 0$. In this case

$$\begin{aligned}
\varphi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(q^{-i} - bb^*\psi^{-1}q^{i-d-1})(q^i\psi - cc^*q^{d-i+1}),\\
\phi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})(q^{-i} - cb^*\psi^{-1}q^{i-d-1})(q^i\psi - bc^*q^{d-i+1}),
\end{aligned}$$

where $\psi \in \overline{\mathbb{F}}$ is a solution to

$$\psi + bb^*cc^*\psi^{-1} = t.$$

Proposition 9.9 *With reference to Assumption 9.1, suppose Φ is a Leonard system and let $\hat{P}$ denote the corresponding normalized Drinfel'd polynomial.*

(i) *Assume* $bb^*cc^* \neq 0$. *Then the roots of* $\hat{P}$ *are*

$$\psi q^{d+1-2i} + bb^*cc^*\psi^{-1}q^{2i-d-1} \qquad\qquad (1 \leq i \leq d),$$

where ψ *is from Lemma 9.8.*

(ii) *Assume* $bb^*cc^* = 0$. *Then the roots of* $\hat{P}$ *are*

$$tq^{d+1-2i} \qquad\qquad (1 \leq i \leq d),$$

where t *is from Lemma 9.8.*

Proof: Assume $d \geq 1$ to avoid trivialities. Let $\{\zeta_i\}_{i=0}^d$ denote the split sequence of Φ from Section 6, and let $\{\varphi_i\}_{i=1}^d$ denote the first split sequence of Φ. By Note 6.1, $\zeta_i = \varphi_1\varphi_2\cdots\varphi_i$ for $0 \leq i \leq d$. By this and (28), $\hat{P}$ is equal to $\varphi_1\varphi_2\cdots\varphi_d$ times

$$\sum_{n=0}^{d} \frac{\hat{p}_d\hat{p}_{d-1}\cdots\hat{p}_{d-n+1}}{\varphi_d\varphi_{d-1}\cdots\varphi_{d-n+1}}. \tag{31}$$

The denominators in (31) are nonzero, since each of $\varphi_1, \varphi_2, \ldots, \varphi_d$ is nonzero by [26, Theorem 1.9].
(i) We evaluate (31) using (26) and the data in Lemma 9.8. This calculation shows that (31) is equal to

$$\sum_{n=0}^{d} \frac{(q^{-2d};q^2)_n(x/bb^*;q^2)_n(cc^*/x;q^2)_n q^{2n}}{(q^{1-d}\psi/bb^*;q^2)_n(q^{1-d}cc^*/\psi;q^2)_n(q^2;q^2)_n},$$

where $\lambda = x + bb^*cc^*x^{-1}$. Basic hypergeometric series are defined in [4, p. 4]. By that definition the above sum is the basic hypergeometric series

$${}_3\phi_2\left[\begin{matrix} q^{-2d},\ x/bb^*,\ cc^*/x \\ q^{1-d}\psi/bb^*,\ \ q^{1-d}cc^*/\psi \end{matrix}\ ;\ q^2,\ q^2\right]. \tag{32}$$

By the q-Saalschütz formula [4, p. 355] the sum (32) is equal to

$$\frac{(q^{1-d}x/\psi;q^2)_d(q^{1-d}bb^*cc^*/\psi x;q^2)_d}{(q^{1-d}bb^*/\psi;q^2)_d(q^{1-d}cc/\psi;q^2)_d}. \tag{33}$$

By (7) the numerator in (33) is equal to

$$\prod_{i=1}^{d}(1 - q^{2i-d-1}x/\psi)(1 - q^{2i-d-1}bb^*cc^*/\psi x). \tag{34}$$

For $1 \leq i \leq d$ the i-factor in (34) is equal to $\psi^{-1}q^{2i-d-1}$ times

$$\psi q^{d+1-2i} + bb^*cc^*\psi^{-1}q^{2i-d-1} - \lambda.$$

Therefore the numerator in (33) is a nonzero scalar multiple of

$$\prod_{i=1}^{d}(\psi q^{d+1-2i} + bb^*cc^*\psi^{-1}q^{2i-d-1} - \lambda).$$

The result follows.
(ii) Replacing q by q^{-1} if necessary, we may assume without loss that $bb^* = 0$. For the moment further assume that the scalar t from Lemma 9.8 is nonzero. Proceeding as in (i) above, we find that (31) is equal to

$${}_2\phi_1\left[\begin{matrix} q^{-2d},\ cc^*/\lambda \\ q^{1-d}cc^*/t \end{matrix}\ ;\ q^2,\ \frac{q^{d+1}\lambda}{t}\right],$$

which is equal to

$$\frac{(q^{1-d}\lambda/t;q^2)_d}{(q^{1-d}cc^*/t;q^2)_d} \tag{35}$$

by the q-Chu-Vandermonde formula [4, p. 354]. By (7) the numerator in (35) is a nonzero scalar multiple of

$$\prod_{i=1}^{d}(tq^{d+1-2i} - \lambda),$$

giving the result for the case $t \neq 0$. Next assume $t = 0$. Then $cc^* \neq 0$; otherwise $\varphi_1 = 0$. In this case (31) is equal to

$${}_2\phi_1\left[\begin{matrix} q^{2d},\ \lambda/cc^* \\ 0 \end{matrix}\ ;\ q^{-2},\ q^{-2}\right],$$

which is equal to $(\lambda/cc^*)^d$ by another version of the q-Chu-Vandermonde formula [4, p. 354]. The result follows for the case $t = 0$, and the proof is complete. □

10 The normalized Drinfel'd polynomial for type II

In this section we introduce the normalized Drinfel'd polynomial for a sharp tridiagonal system of type II.

Assumption 10.1 Let Φ denote a sharp tridiagonal system over $\mathbb{F}$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$. Assume Φ has type II and let the scalars a, b, c, a^*, b^*, c^* be as in Lemma 4.10.

The following definition is motivated by Lemma 4.14.

Definition 10.2 With reference to Assumption 10.1, for $1 \leq i \leq d$ define $\hat{p}_i \in \mathbb{F}[\lambda]$ by

$$\hat{p}_i = i^2(bb^*/2 + (bc^* + cb^*)(d-i) + cc^*(d-i)^2 - \lambda). \tag{36}$$

By Lemma 4.14,

$$\hat{p}_i(-bb^*/2) = (\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i). \tag{37}$$

Definition 10.3 With reference to Assumption 10.1, define $\hat{P} \in \mathbb{F}[\lambda]$ by

$$\hat{P} = \sum_{i=0}^{d} \zeta_i \hat{p}_{i+1}\hat{p}_{i+2}\cdots\hat{p}_d, \tag{38}$$

where $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ and the $\hat{p}_i$ are from Definition 10.2. We call $\hat{P}$ the *normalized Drinfel'd polynomial* for Φ.

Lemma 10.4 *With reference to Assumption 10.1, both*

$$\begin{aligned} ubb^*/2 + v &= \theta_0\theta^*_0 + \theta_d\theta^*_d, \\ -ubb^*/2 + v &= \theta_0\theta^*_d + \theta_d\theta^*_0, \end{aligned}$$

where

$$\begin{aligned} u &= d^2, \\ v &= 2aa^*. \end{aligned}$$

Proof: Use Lemma 4.10. □

Theorem 10.5 *With reference to Assumption 10.1,*

$$\hat{p}_i(\lambda) = p_i(u\lambda + v) \qquad (1 \leq i \leq d), \tag{39}$$

$$\hat{P}(\lambda) = P(u\lambda + v), \tag{40}$$

where u, v are from Lemma 10.4.

Proof: Each side of (39) is a polynomial in $\mathbb{F}[\lambda]$ with degree 1. On each side of (39) the polynomial has λ coefficient $-i^2$. Moreover at $\lambda = -bb^*/2$ this polynomial takes the value $(\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i)$. We have verified (39) and (40) follows. □

Proposition 10.6 *With reference to Assumption 10.1 and Definition 10.3, the polynomial $\hat{P}$ is D_4-invariant.*

Proof: Follows from Proposition 7.15 and (40). □

Proposition 10.7 *With reference to Assumption 10.1 and Definition 10.3, the following* (i), (ii) *hold.*

(i) $\hat{P}(bb^*/2) = \zeta_d$;

(ii) $\hat{P}(-bb^*/2) = \zeta_d^{\Downarrow}$.

Proof: Combine Proposition 7.16, Lemma 10.4, and (40). □

We now consider the normalized Drinfel'd polynomial for a Leonard system of type II.

Lemma 10.8 [15, Theorem 7.1] *With reference to Assumption 10.1, suppose Φ is a Leonard system. Let $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denote the corresponding first (resp. second) split sequence, from [26, Section 3]. Then there exists $t \in \overline{\mathbb{F}}$ such that for $1 \leq i \leq d$,*

$$\begin{aligned} \varphi_i &= i(d-i+1)\big(t - bb^*/2 + (bc^* + cb^*)(i - \tfrac{d+1}{2}) - cc^*(i - \tfrac{d+1}{2})^2\big), \\ \phi_i &= i(d-i+1)\big(t + bb^*/2 + (cb^* - bc^*)(i - \tfrac{d+1}{2}) - cc^*(i - \tfrac{d+1}{2})^2\big). \end{aligned}$$

The above expressions can be factored more completely if $cc^* \neq 0$. In this case

$$\begin{aligned} \varphi_i &= c^{-1}c^{*-1}i(d-i+1)\big(\tfrac{\psi+bc^*+cb^*}{2} - cc^*(i - \tfrac{d+1}{2})\big)\big(\tfrac{\psi-bc^*-cb^*}{2} + cc^*(i - \tfrac{d+1}{2})\big), \\ \phi_i &= c^{-1}c^{*-1}i(d-i+1)\big(\tfrac{\psi-bc^*+cb^*}{2} - cc^*(i - \tfrac{d+1}{2})\big)\big(\tfrac{\psi+bc^*-cb^*}{2} + cc^*(i - \tfrac{d+1}{2})\big), \end{aligned}$$

where $\psi \in \overline{\mathbb{F}}$ is a solution to

$$\psi^2 = 4tcc^* + b^2c^{*2} + b^{*2}c^2.$$

Proposition 10.9 *With reference to Assumption 10.1, suppose Φ is a Leonard system and let $\hat{P}$ denote the corresponding normalized Drinfel'd polynomial.*

(i) *Assume $cc^* \neq 0$. Then the roots of $\hat{P}$ are*

$$t + \psi(i - \tfrac{d+1}{2}) + cc^*(i - \tfrac{d+1}{2})^2 \qquad (1 \leq i \leq d),$$

where ψ, t are from Lemma 10.8.

(ii) *Assume $cc^* = 0$. Then the roots of $\hat{P}$ are*

$$t + (bc^* + cb^*)(i - \tfrac{d+1}{2}) \qquad (1 \leq i \leq d),$$

where t is from Lemma 10.8.

Proof: We begin as in the proof of Proposition 9.9.
(i) We evaluate (31) using (36) and the data in Lemma 10.8. The result is that (31) is equal to

$$\sum_{n=0}^{d} \frac{(-d)_n \left(\frac{bc^*+cb^*+x}{2cc^*}\right)_n \left(\frac{bc^*+cb^*-x}{2cc^*}\right)_n}{\left(\frac{bc^*+cb^*+\psi}{2cc^*} + \frac{1-d}{2}\right)_n \left(\frac{bc^*+cb^*-\psi}{2cc^*} + \frac{1-d}{2}\right)_n n!},$$

where $x^2 = 4\lambda cc^* + b^2c^{*2} + b^{*2}c^2$ and $a_n = a(a+1)\cdots(a+n-1)$. Hypergeometric series are defined in [4, p. 3]. By that definition the above sum is the hypergeometric series

$$ {}_3F_2\left[\begin{matrix} -d, & \frac{bc^*+cb^*+x}{2cc^*}, & \frac{bc^*+cb^*-x}{2cc^*} \\ \frac{bc^*+cb^*+\psi}{2cc^*} + \frac{1-d}{2}, & \frac{bc^*+cb^*-\psi}{2cc^*} + \frac{1-d}{2} & \end{matrix} ; 1\right]. \tag{41} $$

By the Saalschütz formula [4, p. 17] the sum (41) is equal to

$$ \frac{\left(\frac{\psi-x}{2cc^*} + \frac{1-d}{2}\right)_d \left(\frac{\psi+x}{2cc^*} + \frac{1-d}{2}\right)_d}{\left(\frac{\psi+bc^*+cb^*}{2cc^*} + \frac{1-d}{2}\right)_d \left(\frac{\psi-bc^*-cb^*}{2cc^*} + \frac{1-d}{2}\right)_d}. \tag{42} $$

The numerator in (42) is equal to

$$ \prod_{i=1}^{d} \left(\frac{\psi-x}{2cc^*} + i - \frac{d+1}{2}\right)\left(\frac{\psi+x}{2cc^*} + i - \frac{d+1}{2}\right). \tag{43} $$

For $1 \le i \le d$ the i-factor in (43) is equal to $(cc^*)^{-1}$ times

$$ t + \psi(i - \tfrac{d+1}{2}) + cc^*(i - \tfrac{d+1}{2})^2 - \lambda. $$

Therefore the numerator in (42) is a nonzero scalar multiple of

$$ \prod_{i=1}^{d} \left(t + \psi(i - \tfrac{d+1}{2}) + cc^*(i - \tfrac{d+1}{2})^2 - \lambda\right). $$

The result follows.
(ii) Replacing Φ by Φ^* if necessary, we may assume without loss that $c = 0$. For the moment assume further that $c^* \neq 0$. Then (31) is equal to

$$ {}_2F_1\left[\begin{matrix} -d, & \frac{b^*}{2c^*} - \frac{\lambda}{bc^*} \\ \frac{b^*}{2c^*} - \frac{t}{bc^*} + \frac{1-d}{2} & \end{matrix} ; 1\right], $$

which is equal to

$$ \frac{\left(\frac{\lambda-t}{bc^*} + \frac{1-d}{2}\right)_d}{\left(\frac{b^*}{2c^*} - \frac{t}{bc^*} + \frac{1-d}{2}\right)_d} \tag{44} $$

by the Chu-Vandermonde formula [4, p. 2]. The numerator in (44) is a nonzero scalar multiple of

$$ \prod_{i=1}^{d} \left(t + bc^*(i - \tfrac{d+1}{2}) - \lambda\right), $$

giving the result for the case $c^* \neq 0$. Next assume $c^* = 0$. Then $bb^* \neq 2t$; otherwise $\varphi_1 = 0$. In this case (31) is equal to

$$ {}_1F_0\left[\begin{matrix} -d \\ \ \end{matrix} ; \frac{bb^* - 2\lambda}{bb^* - 2t}\right], $$

which is equal to $2^d(\lambda - t)^d(bb^* - 2t)^{-d}$ by the binomial theorem. The result follows for the case $c^* = 0$, and the proof is complete. $\square$

11 The normalized Drinfel'd polynomial for type III

In this section we introduce the normalized Drinfel'd polynomial for a sharp tridiagonal system of type III.

For a sharp tridiagonal system of type III$^+$ we define the normalized Drinfel'd polynomial $\hat{P}$ to be the Drinfel'd polynomial P from Definition 8.1. For the rest of this section we focus on type III$^-$.

Assumption 11.1 Let Φ denote a sharp tridiagonal system over $\mathbb{F}$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$. Assume Φ has type III$^-$ and let the scalars a, b, c, a^*, b^*, c^* be as in Lemma 4.15.

The following definition is motivated by Lemma 4.20.

Definition 11.2 With reference to Assumption 11.1, for $1 \le i \le d$ define $\hat{p}_i \in \mathbb{F}[\lambda]$ by

$$\hat{p}_i = \begin{cases} cc^* i^2 & \text{if } i \text{ is even,} \\ 2bb^* + 2(bc^* + cb^*)(i-d) + cc^*(i-d)^2 - \lambda & \text{if } i \text{ is odd,} \end{cases} \tag{45}$$

By Lemma 4.20,

$$\hat{p}_i(-2bb^*) = (\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i). \tag{46}$$

Definition 11.3 With reference to Assumption 11.1, define $\hat{P} \in \mathbb{F}[\lambda]$ by

$$\hat{P} = \sum_{i=0}^{d} \zeta_i \hat{p}_{i+1} \hat{p}_{i+2} \cdots \hat{p}_d, \tag{47}$$

where $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ and the $\hat{p}_i$ are from Definition 11.2. We call $\hat{P}$ the *normalized Drinfel'd polynomial* for Φ. We note that $\hat{P}$ has degree exactly $(d+1)/2$.

Lemma 11.4 *With reference to Assumption 11.1, both*

$$\begin{aligned} 2ubb^* + v &= \theta_0\theta^*_0 + \theta_d\theta^*_d, \\ -2ubb^* + v &= \theta_0\theta^*_d + \theta_d\theta^*_0, \end{aligned}$$

where

$$\begin{aligned} u &= 1, \\ v &= (2a - cd)(2a^* - c^*d)/2. \end{aligned}$$

Proof: Use Lemma 4.15. □

Theorem 11.5 *With reference to Assumption 11.1,*

$$\begin{aligned} \hat{p}_i(\lambda) &= p_i(u\lambda + v) \qquad\qquad (1 \le i \le d), && (48)\\ \hat{P}(\lambda) &= P(u\lambda + v), && (49) \end{aligned}$$

where u, v are from Lemma 11.4.

Proof: Each side of (48) is a polynomial in $\mathbb{F}[\lambda]$ with degree at most 1. On each side of (48) the polynomial has λ coefficient 0 (if i is even) and -1 (if i is odd). Moreover at $\lambda = -2bb^*$ this polynomial takes the value $(\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i)$. Now (48) is true and (49) follows. $\square$

Proposition 11.6 *With reference to Assumption 11.1 and Definition 11.3, the polynomial $\hat{P}$ is D_4-invariant.*

Proof: Follows from Theorem 7.15 and (49). $\square$

Proposition 11.7 *With reference to Assumption 11.1 and Definition 11.3, the following* (i), (ii) *hold.*

(i) $\hat{P}(2bb^*) = \zeta_d$;

(ii) $\hat{P}(-2bb^*) = \zeta^{\Downarrow}_d$.

Proof: Combine Proposition 7.16, Lemma 11.4, and (49). $\square$

We now consider the normalized Drinfel'd polynomial for a Leonard system of type III$^-$.

Lemma 11.8 [15, Theorem 9.1] *With reference to Assumption 11.1, suppose Φ is a Leonard system. Let $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denote the corresponding first (resp. second) split sequence, from [26, Section 3]. Then there exists $t \in \overline{\mathbb{F}}$ such that for $1 \le i \le d$,*

$$\varphi_i = \begin{cases} cc^* i(d-i+1) & \text{if } i \text{ is even,}\\ t - 2bb^* - 2(bc^* + cb^*)(i - \frac{d+1}{2}) - cc^*(i - \frac{d+1}{2})^2 & \text{if } i \text{ is odd,} \end{cases}$$

$$\phi_i = \begin{cases} cc^* i(d-i+1) & \text{if } i \text{ is even,}\\ t + 2bb^* + 2(bc^* - cb^*)(i - \frac{d+1}{2}) - cc^*(i - \frac{d+1}{2})^2 & \text{if } i \text{ is odd.} \end{cases}$$

For i odd we have

$$\begin{aligned} \varphi_i &= c^{-1}c^{*-1}\big(\psi - bc^* - cb^* - cc^*(i - \tfrac{d+1}{2})\big)\big(\psi + bc^* + cb^* + cc^*(i - \tfrac{d+1}{2})\big),\\ \phi_i &= c^{-1}c^{*-1}\big(\psi + bc^* - cb^* - cc^*(i - \tfrac{d+1}{2})\big)\big(\psi - bc^* + cb^* + cc^*(i - \tfrac{d+1}{2})\big), \end{aligned}$$

where $\psi \in \overline{\mathbb{F}}$ is a solution to

$$\psi^2 = tcc^* + b^2c^{*2} + b^{*2}c^2.$$

Proposition 11.9 *With reference to Assumption 11.1, suppose Φ is a Leonard system and let $\hat{P}$ denote the corresponding normalized Drinfel'd polynomial. Then the roots of $\hat{P}$ are*

$$t + 2\psi(i - \tfrac{d+1}{2}) + cc^*(i - \tfrac{d+1}{2})^2 \qquad (1 \le i \le d,\ i \text{ odd}),$$

where ψ, t are from Lemma 11.8.

Proof: Abbreviate $N = (d+1)/2$. We begin as in the proof of Proposition 9.9. We evaluate (31) using (45) and the data in Lemma 11.8. The result is a hypergeometric series

$$ {}_3F_2\left[\begin{matrix} -N, & \frac{x-bc^*-cb^*}{2cc^*}, & -\frac{x+bc^*+cb^*}{2cc^*} \\ \frac{1-d}{4} - \frac{bc^*+cb^*-\psi}{2cc^*}, & \frac{1-d}{4} - \frac{bc^*+cb^*+\psi}{2cc^*} \end{matrix} ; 1\right], \qquad (50)$$

where $x^2 = \lambda cc^* + b^2c^{*2} + b^{*2}c^2$. The terms in the series (31) are related to the terms in the series (50) as follows. For even n $(0 < n < d)$ the n-summand in (31) plus the $(n-1)$-summand in (31) is equal to the $(n/2)$-summand in (50). The 0-summand in (31) is equal to the 0-summand in (50), and the d-summand in (31) is equal to the N-summand in (50). By the Saalschütz formula [4, p. 17] the sum (50) is equal to

$$\frac{\left(\frac{\psi-x}{2cc^*} + \frac{1-d}{4}\right)_N \left(\frac{\psi+x}{2cc^*} + \frac{1-d}{4}\right)_N}{\left(\frac{\psi+bc^*+cb^*}{2cc^*} + \frac{1-d}{4}\right)_N \left(\frac{\psi-bc^*-cb^*}{2cc^*} + \frac{1-d}{4}\right)_N}. \qquad (51)$$

The numerator in (51) can be expressed as

$$\prod_{\substack{1\le i\le d \\ i \text{ odd}}} \tfrac{1}{4}\left(\tfrac{\psi-x}{cc^*} + i - \tfrac{d+1}{2}\right)\left(\tfrac{\psi+x}{cc^*} + i - \tfrac{d+1}{2}\right). \qquad (52)$$

For odd i $(1 \le i \le d)$ the i-factor in (52) is equal to $(4cc^*)^{-1}$ times

$$t + 2\psi(i - \tfrac{d+1}{2}) + cc^*(i - \tfrac{d+1}{2})^2 - \lambda.$$

Therefore the numerator in (51) is a nonzero scalar multiple of

$$\prod_{\substack{1\le i\le d \\ i \text{ odd}}} \left(t + 2\psi(i - \tfrac{d+1}{2}) + cc^*(i - \tfrac{d+1}{2})^2 - \lambda\right).$$

The result follows. $\square$

12 The normalized Drinfel'd polynomial for type IV

In this section we discuss the normalized Drinfel'd polynomial for a sharp tridiagonal system of type IV. For this type it will turn out that the normalized Drinfel'd polynomial is the same as the Drinfel'd polynomial, but for notational consistency we will continue the hat notation from Sections 9–11.

Assumption 12.1 Let Φ denote a sharp tridiagonal system over $\mathbb{F}$ that has type IV. Let $\{\theta_i\}_{i=0}^3$ (resp. $\{\theta^*_i\}_{i=0}^3$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ. Let the scalars a, b, c, a^*, b^*, c^* be as in Lemma 4.21.

Definition 12.2 With reference to Assumption 12.1, define $\hat{p}_i \in \mathbb{F}[\lambda]$ by

$$\begin{aligned}
\hat{p}_1 &= ab^* + ba^* + (a+b+c)(a^*+b^*+c^*) + \lambda, \\
\hat{p}_2 &= cc^*, \\
\hat{p}_3 &= ab^* + ba^* + (a+b)(a^*+b^*) + \lambda.
\end{aligned}$$

By Lemma 4.23,

$$\hat{p}_i(ab^* + ba^*) = (\theta_0 - \theta_i)(\theta^*_0 - \theta^*_i). \tag{53}$$

Definition 12.3 With reference to Assumption 12.1, define

$$\hat{P} = \hat{p}_1\hat{p}_2\hat{p}_3 + \zeta_1\hat{p}_2\hat{p}_3 + \zeta_2\hat{p}_3 + \zeta_3, \tag{54}$$

where $\{\zeta_i\}_{i=0}^3$ is the split sequence of Φ and the $\hat{p}_i$ are from Definition 12.2. We call $\hat{P}$ the *normalized Drinfel'd polynomial* for Φ. We note that $\hat{P}$ has degree exactly 2.

Lemma 12.4 *With reference to Assumption 12.1,*

$$\hat{p}_i = p_i \qquad (1 \le i \le 3), \tag{55}$$

$$\hat{P} = P. \tag{56}$$

Proof: By Definition 7.1 and since $\mathrm{Char}(\mathbb{F}) = 2$,

$$\begin{aligned}
p_1 &= \theta_0\theta^*_3 + \theta_3\theta^*_0 + \lambda + (\theta_0 - \theta_1)(\theta^*_0 - \theta^*_1), \\
p_2 &= (\theta_0 - \theta_2)(\theta^*_0 - \theta^*_2), \\
p_3 &= \theta_0\theta^*_3 + \theta_3\theta^*_0 + \lambda + (\theta_0 - \theta_3)(\theta^*_0 - \theta^*_3).
\end{aligned}$$

Evaluating these lines using Lemma 4.21, Lemma 4.23 and comparing the result with Definition 12.2, we get (55). Line (56) follows in view of (19) and (54). □

Proposition 12.5 *With reference to Assumption 12.1 and Definition 12.3, the polynomial* $\hat{P}$ *is* D_4*-invariant.*

Proof: Clear from Theorem 7.15 and Lemma 12.4. □

Proposition 12.6 *With reference to Assumption 12.1 and Definition 12.3, the following* (i), (ii) *hold.*

(i) $\hat{P}(aa^* + bb^*) = \zeta_3$;

(ii) $\hat{P}(ab^* + ba^*) = \zeta_3^{\Downarrow}$.

Proof: Evaluate Proposition 7.16 using Lemma 4.21 and (56). $\square$

We now consider the normalized Drinfel'd polynomial for a Leonard system of type IV.

Lemma 12.7 [15, Theorem 10.1] *With reference to Assumption 12.1, suppose Φ is a Leonard system. Let $\{\varphi_i\}_{i=1}^3$ (resp. $\{\phi_i\}_{i=1}^3$) denote the corresponding first (resp. second) split sequence, from [26, Section 3]. Then there exists $\varphi \in \mathbb{F}$ such that*

$$\varphi_1 = \varphi, \qquad \varphi_2 = cc^*, \qquad \varphi_3 = \varphi + (a+b)c^* + c(a^*+b^*),$$
$$\phi_1 = \varphi + (a+b)(a^*+b^*+c^*), \qquad \phi_2 = cc^*, \qquad \phi_3 = \varphi + (a+b+c)(a^*+b^*).$$

Proposition 12.8 *With reference to Assumption 12.1, suppose Φ is a Leonard system and let $\hat{P}$ denote the corresponding normalized Drinfel'd polynomial. Then $\hat{P}$ is cc^* times*

$$\begin{aligned}&(\lambda + ab^* + ba^*)^2 \; + \; (\lambda + ab^* + ba^*)(ac^* + bc^* + ca^* + cb^* + cc^*)\\ &\qquad + \varphi^2 \; + \; \varphi(ac^* + bc^* + ca^* + cb^*) \; + \; (a+b)(a^*+b^*)(a+b+c)(a^*+b^*+c^*).\end{aligned}$$

Proof: Evaluate (54) using Definition 12.2, Lemma 12.7, and $\zeta_1 = \varphi_1$, $\zeta_2 = \varphi_1\varphi_2$, $\zeta_3 = \varphi_1\varphi_2\varphi_3$. $\square$

Referring to Proposition 12.8, we caution the reader that since $\mathrm{Char}(\mathbb{F}) = 2$ the roots of $\hat{P}$ cannot be obtained using the quadratic formula. To get these roots we proceed as follows.

Lemma 12.9 *With reference to Assumption 12.1, suppose Φ is a Leonard system. Let $\{\varphi_i\}_{i=1}^3$ (resp. $\{\phi_i\}_{i=1}^3$) denote the corresponding first (resp. second) split sequence. Then there exists $\psi \in \overline{\mathbb{F}}$ such that*

$$\begin{aligned}cc^*\varphi_1 &= (ac^* + a^*c + cc^*\psi)(bc^* + b^*c + cc^* + cc^*\psi),\\ cc^*\varphi_3 &= (bc^* + b^*c + cc^*\psi)(ac^* + a^*c + cc^* + cc^*\psi),\\ cc^*\phi_1 &= (bc^* + a^*c + cc^*\psi)(ac^* + b^*c + cc^* + cc^*\psi),\\ cc^*\phi_3 &= (ac^* + b^*c + cc^*\psi)(bc^* + a^*c + cc^* + cc^*\psi).\end{aligned}$$

Proof: Since $cc^* \neq 0$ and since $\overline{\mathbb{F}}$ is algebraically closed there exists $\psi \in \overline{\mathbb{F}}$ that satisfies the first of the four equations above. The remaining three equations follow in view of the data in Lemma 12.7. $\square$

We comment on the uniqueness of the scalar ψ in Lemma 12.9.

Note 12.10 If $\psi \in \overline{\mathbb{F}}$ satisfies the four equations in Lemma 12.9, then

$$\psi + \frac{a+b}{c} + \frac{a^*+b^*}{c^*} + 1$$

satisfies these equations and no other scalar in $\overline{\mathbb{F}}$ satisfies these equations.

Theorem 12.11 *With reference to Assumption 12.1, suppose Φ is a Leonard system and let $\hat{P}$ denote the corresponding normalized Drinfel'd polynomial. Then the roots of $\hat{P}$ are*

$$ab^* + ba^* + (cc^*\psi + ac^* + b^*c)(cc^*\psi + a^*c + bc^*)c^{-1}c^{*-1},$$
$$ab^* + ba^* + (cc^*\psi + ac^* + b^*c + cc^*)(cc^*\psi + a^*c + bc^* + cc^*)c^{-1}c^{*-1},$$

where ψ is from Lemma 12.9.

Proof: Denote the above expressions by z_1, z_2. One finds $\hat{P}(z_1) = 0$, $\hat{P}(z_2) = 0$ using Proposition 12.8 and Lemma 12.9. $\square$

Combining Propositions 9.6, 10.6, 11.6, 12.5 and the second paragraph in Section 11, we obtain the following theorem.

Theorem 12.12 *Let Φ denote a sharp tridiagonal system and let $\hat{P}$ denote the corresponding normalized Drinfel'd polynomial. Then $\hat{P}$ is D_4-invariant.*

Definition 12.13 Let A, A^* denote a sharp tridiagonal pair. By the *normalized Drinfel'd polynomial* for A, A^* we mean the normalized Drinfel'd polynomial for an associated tridiagonal system. By construction A, A^* and A^*, A have the same normalized Drinfel'd polynomial.

13 Why P is called the Drinfel'd polynomial

Let A, A^* denote a sharp tridiagonal pair. Earlier in the paper, we associated with this pair a polynomial P called the Drinfel'd polynomial. In this section we justify the name by relating P to the classical Drinfel'd polynomial from the theory of Lie algebras and quantum groups.

Throughout this section we assume that the field $\mathbb{F}$ is algebraically closed with characteristic zero.

Definition 13.1 [12, Section 1] Let A, A^* denote a tridiagonal pair over $\mathbb{F}$ that has diameter d. This pair is said to have *Krawtchouk type* whenever the sequence $\{d-2i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A and a standard ordering of the eigenvalues of A^*.

Let A, A^* denote a tridiagonal pair on V that has Krawtchouk type. By [5, Theorem 1.8] the pair A, A^* induces on V a module structure for the 3-point $\mathfrak{sl}_2$ loop algebra [6, Definition 1.1]. Associated with this module is a Drinfel'd polynomial [11, Definition 9.13], [12, Lemma 13.2] which we denote by P_{A,A^*}. In the notation of the present paper P_{A,A^*} looks as follows.

Definition 13.2 [12, Definition 13.1] Let A, A^* denote a tridiagonal pair over $\mathbb{F}$ that has Krawtchouk type. Define $P_{A,A^*} \in \mathbb{F}[\lambda]$ by

$$P_{A,A^*} = \sum_{i=0}^{d} \frac{(-1)^i \zeta_i \lambda^i}{(i!)^2 4^i}, \tag{57}$$

where $\{\zeta_i\}_{i=0}^d$ is the split sequence of A, A^* associated with the standard ordering $\{d-2i\}_{i=0}^d$ (resp. $\{2i-d\}_{i=0}^d$) of the eigenvalues of A (resp. A^*).

Theorem 13.3 *Let A, A^* denote a tridiagonal pair over $\mathbb{F}$ that has Krawtchouk type, and let $\hat{P}$ denote the associated normalized Drinfel'd polynomial from Definition 12.13. Then*

$$\hat{P}(\lambda) = (-1)^d (d!)^2 (\lambda+2)^d P_{A,A^*}(4(\lambda+2)^{-1}), \tag{58}$$

where P_{A,A^*} is from Definition 13.2 and d is the diameter of A, A^*.

Proof: To describe $\hat{P}$ we associate with A, A^* a tridiagonal system. For $0 \leq i \leq d$ let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) associated with the eigenvalue $d-2i$ (resp. $2i-d$). Then $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a tridiagonal system with eigenvalue sequence $\{d-2i\}_{i=0}^d$ and dual eigenvalue sequence $\{2i-d\}_{i=0}^d$. By Definition 4.5 Φ is type II, and the equations of Lemma 4.10 are satisfied by

$$a = 0, \quad a^* = 0, \quad b = -2, \quad b^* = 2, \quad c = 0, \quad c^* = 0.$$

Evaluating (36) using this we find $\hat{p}_i(\lambda) = -i^2(\lambda+2)$ for $1 \leq i \leq d$. Now using (38),

$$\hat{P}(\lambda) = (-1)^d (d!)^2 (\lambda+2)^d \sum_{i=0}^{d} \frac{(-1)^i \zeta_i}{(i!)^2 (\lambda+2)^i}. \tag{59}$$

Comparing (57) and (59) we obtain (58). $\square$

For the rest of this section fix a nonzero $q \in \mathbb{F}$ that is not a root of 1. For all integers $n \geq 0$ define

$$\begin{aligned} [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, \\ [n]_q^! &= [n]_q [n-1]_q \cdots [1]_q. \end{aligned}$$

We interpret $[0]_q^! = 1$.

Definition 13.4 [10, Definition 2.6] Let A, A^* denote a tridiagonal pair over $\mathbb{F}$ that has diameter d. Then A, A^* is called *q-geometric* whenever the sequence $\{q^{d-2i}\}_{i=0}^d$ is a standard ordering of the eigenvalues of A and a standard ordering of the eigenvalues of A^*.

Let A, A^* denote a q-geometric tridiagonal pair on V. By [9, Theorem 3.3, Theorem 13.1] the pair A, A^* induces on V a module structure for the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. Associated with this module is a Drinfel'd polynomial [10, Definition 4.2] which we will denote by P_{A,A^*}. In the notation of the present paper P_{A,A^*} looks as follows.

Definition 13.5 [10, Definition 4.2] Let A, A^* denote a q-geometric tridiagonal pair over $\mathbb{F}$. Define $P_{A,A^*} \in \mathbb{F}[\lambda]$ by

$$P_{A,A^*} = \sum_{i=0}^{d} \frac{(-1)^i \zeta_i q^i \lambda^i}{([i]_q^!)^2}, \tag{60}$$

where $\{\zeta_i\}_{i=0}^d$ is the split sequence of A, A^* associated with the standard ordering $\{q^{2i-d}\}_{i=0}^d$ (resp. $\{q^{d-2i}\}_{i=0}^d$) of the eigenvalues of A (resp. A^*).

Theorem 13.6 *Let A, A^* denote a q-geometric tridiagonal pair over $\mathbb{F}$, and let $\hat{P}$ denote the associated normalized Drinfel'd polynomial from Definition 12.13. Then*

$$\hat{P}(\lambda) = (-1)^d([d]^!_q)^2(q - q^{-1})^{2d}\lambda^d P_{A,A^*}(\lambda^{-1}q^{-1}(q - q^{-1})^{-2}), \tag{61}$$

where P_{A,A^*} is from Definition 13.5 and d is the diameter of A, A^*.

Proof: To describe $\hat{P}$ we associate with A, A^* a tridiagonal system. For $0 \leq i \leq d$ let E_i (resp. E^*_i) denote the primitive idempotent of A (resp. A^*) associated with the eigenvalue q^{2i-d} (resp. q^{d-2i}). Then $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ is a tridiagonal system with eigenvalue sequence $\{q^{2i-d}\}_{i=0}^d$ and dual eigenvalue sequence $\{q^{d-2i}\}_{i=0}^d$. Referring to Definition 4.3, we have $\beta = q^2 + q^{-2}$ and q is not a root of unity so Φ is type I. Without loss we may take the scalar q from Lemma 4.6 to be the present q that we fixed above Definition 13.4. The equations of Lemma 4.6 are satisfied by

$$a = 0, \quad a^* = 0, \quad b = 1, \quad b^* = 0, \quad c = 0, \quad c^* = 1.$$

Evaluating (26) using this we find $\hat{p}_i = -(q^i - q^{-i})^2\lambda$ for $1 \leq i \leq d$. Now using (28),

$$\hat{P}(\lambda) = (-1)^d([d]^!_q)^2(q - q^{-1})^{2d}\lambda^d \sum_{i=0}^{d} \frac{(-1)^i\zeta_i}{(q - q^{-1})^{2i}([i]^!_q)^2\lambda^i}. \tag{62}$$

Comparing (60) and (62) we obtain (61). $\square$

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